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Conformal identities for invariant second-order variational problems depending on a covariant vector field

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Abstract. In this paper, necessary and sufficient conditions for the conformal invariance of a multiple integral variational problem whose Lagrangian depends upon second-order derivatives of a covariant vector field are obtained. These conditions take the form of differential identities involving the Lagrangian, its derivatives, and the infinitesimal generators of the special conformal group; they differ from the classical Noether identities in that they involve only second-order derivatives of the field, not fourth-order derivatives. The conditions are not conservation laws, but rather identities which provide a practical test for invariance which, if established, can lead to conservation laws via the Noether theorem. Finally, an application to 'generalised electrodynamics' is given.

1. Introduction

It is well known that if a variational integral is invariant under an r-parameter local Lie group of transformations, then r combinations of the Euler–Lagrange expressions can be written as divergences; this is the classical Noether theorem on invariant variational problems (see Noether 1918, also Tavel 1971 for a translation in English, Logan 1977). Along an extremal, these r identities result in conservation laws for the system described by the governing Euler–Lagrange equations. Recently, Rund (1972) gave a new derivation of the Noether identities based upon a more fundamental set of invariance identities. These identities, which are similar to the Killing equations (see Logan 1975), involve the Lagrangian, its derivatives, and the infinitesimal generators of the transformations, and they contain lower derivatives of the field functions than do the classical identities of Noether. In the special case of the fifteen-parameter special conformal group (see Haantjes 1937, Ohyama 1943, and Fulton *et al* 1962), Bessel-Hagen (1921) was first to investigate the consequences of conformal invariance of the action integral in electrodynamics by applying the Noether theorem to obtain the conservation laws. With respect to the identities obtained by Rund, Logan (1974) showed how those identities along with the assumption of invariance can lead to a characterisation of conformally invariant variational problems for scalar and covariant vector fields.

The results mentioned above are for the case when the Lagrangian depends only upon the field functions and their first derivatives. For second-order problems, i.e. when the Lagrangian also depends upon second derivatives of the field functions, the Noether theorem and results on conservation laws are again well known (see Barut

1964 or Logan 1977, for example). In this paper, however, we wish to derive the conformal invariance identities for the second-order problem in the covariant vector field case. Hence, the results in this paper can be considered as an application of the general invariance theory for second-order variational problems which has been developed as an extension of Rund's work by Logan and Blakeslee (1975) and Blakeslee and Logan (1976). We shall obtain a necessary and sufficient condition for a second-order variational problem to be invariant under the special conformal group. From a practical point of view, this condition is useful as a test for conformal invariance, as we shall show by determining the non-conformal invariance of 'generalised electrodynamics'.

The role of second-order problems in physical theories, especially continuum mechanics and relativity (see Grässer 1967, Borneas 1963, Logan 1977) is well established. In our case we shall investigate the consequences of imposing invariance on a multiple integral variational problem where the Lagrangian depends upon a covariant vector field and its first and second derivatives. To be more precise, the Lagrangian is a function

$$L: D\times R^n\times R^{nm}\times R^{nm^2}\to R^1,$$

where D is an open region in R^m, which is assumed to be of class C^2 in all of its variables. For convenience, we set

$$\partial_\alpha x_k(t)\equiv\frac{\partial x_k(t)}{\partial t^\alpha},\qquad \partial_{\alpha\beta}x_k(t)\equiv\frac{\partial^2 x_k(t)}{\partial t^\alpha\partial t^\beta}$$

where $x: D\to R^n$ is a C^4 function having components $x_k(t)$, $k=1,\ldots,n$. Here, and in the sequel, lower case Latin indices $j, k, \ldots$ range over $1,\ldots,n$ and lower case Greek indices $\alpha, \beta, \ldots$ range over $1,\ldots,m$. A generic element in the domain of L will be written $(t, x, \partial x, \partial^2 x)$ where $t=(t^1,\ldots,t^m)$, $x=(x_1,\ldots,x_n)$, $\partial x=(\ldots,\partial_\alpha x_k,\ldots)$, and $\partial^2 x=(\ldots,\partial_{\alpha\beta}x_k,\ldots)$. The action integral is then given by

$$J(x)=\int_{G_m} L(t, x(t), \partial x(t), \partial^2 x(t))\,\mathrm{d}t^1\ldots\mathrm{d}t^m \tag{1}$$

where G_m is a compact subset of D.

Our calculations are then based on the following theorem (Blakeslee and Logan 1975).

Theorem 1. A necessary and sufficient condition that (1) be invariant under the r-parameter family of transformations

$$\bar{t}^\alpha=\phi^\alpha(t, x, \epsilon),\qquad \bar{x}_k=\psi_k(t, x, \epsilon),\qquad \epsilon=(\epsilon^1,\ldots,\epsilon^r) \tag{2}$$

is that the following r invariance identities hold true:

$$\frac{\mathrm{d}L}{\mathrm{d}t^\alpha}\tau_s^\alpha+L\frac{\mathrm{d}\tau_s^\alpha}{\mathrm{d}t^\alpha}+\frac{\partial L}{\partial x_k}C_{ks}+\frac{\partial L}{\partial(\partial_\alpha x_k)}\frac{\mathrm{d}}{\mathrm{d}t^\alpha}C_{ks}+\frac{\partial L}{\partial(\partial_{\alpha\beta}x_k)}\frac{\mathrm{d}^2}{\mathrm{d}t^\alpha\,\mathrm{d}t^\beta}C_{ks}=0\qquad(s=1,\ldots,r) \tag{3}$$

where

$$\tau_s^\alpha\equiv\left.\frac{\partial\phi^\alpha}{\partial\epsilon^s}\right|_{\epsilon=0},\qquad \xi_{ks}\equiv\left.\frac{\partial\psi_k}{\partial\epsilon^s}\right|_{\epsilon=0} \tag{4}$$

are the generators of (2) and

$$C_{ks} \equiv \xi_{ks} - \partial_\gamma x_k \tau_s^\gamma. \tag{5}$$

Remark 1. It is of course assumed that the mappings ϕ^α and ψ_k are of class C^4 in each of their $m+n+r$ arguments, and when $\epsilon = 0$ the identity transformation results.

Remark 2. The generators ξ_{ks} of the $x_k \to \bar{x}_k$ transformation are determined from the tensorial character of the field. In the present case we assume that the x_k are components of a covariant vector field, i.e.

$$\bar{x}_\alpha(\bar{t}) = \frac{\partial t^\beta}{\partial \bar{t}^\alpha} x_\beta(t). \tag{6}$$

It then follows that (see Logan 1977)

$$\xi_{\alpha s} = -x_\lambda \frac{\partial \tau_s^\lambda}{\partial t^\alpha}, \tag{7}$$

where τ_s^λ are the generators of the $t^\alpha \to \bar{t}^\alpha$ transformation.

2. Conformal transformations

We adopt the standard space–time coordinates $t^0 = \mathrm{i}ct$, $t^1 = x$, $t^2 = y$, $t^3 = z$ of special relativity. Therefore (see Fulton *et al* 1962) the special conformal transformations can be written explicitly as:

(i) space–time translation (four parameters)

$$\bar{t}^\alpha = t^\alpha + \epsilon^\alpha; \tag{8}$$

(ii) space–time rotations (six parameters)

$$\bar{t}^\alpha = t^\alpha + \omega_\beta^\alpha t^\beta, \qquad \omega_\beta^\alpha = -\omega_\alpha^\beta; \tag{9}$$

(iii) dilation (one parameter)

$$\bar{t}^\alpha = t^\alpha + \gamma t^\alpha; \tag{10}$$

(iv) inversions (four parameters)

$$\bar{t}^\alpha = t^\alpha + (2t^\alpha t_\lambda - t^\nu t_\nu \delta_\lambda^\alpha)\eta^\lambda; \tag{11}$$

where $\epsilon^1, \ldots, \epsilon^4$, ω_2^1, ω_3^1, ω_4^1, ω_3^2, ω_4^2, ω_4^3, γ, η^1, η^2, η^3, and η^4 are the fifteen independent parameters. The generators τ_s^α can be obtained directly by differentiating (8) through (11) according to (4), and consequently the generators $\xi_{\alpha s}$ may be calculated using (7) which follows from the assumption that $x(t)$ is a covariant vector field. We summarise the result in the following lemma.

Lemma. When $x = x(t)$ is a covariant vector field, the generators (4) of the special conformal group (8)–(11) are given by:

(i) translations

$$\tau_\beta^\alpha = \delta_\beta^\alpha, \qquad \xi_{\alpha s} = 0; \tag{12}$$

(ii) rotations

$$\tau^{\alpha}_{\mu\nu} = \delta^{\alpha}_{\mu} t^{\nu} - \delta^{\alpha}_{\nu} t^{\mu}, \qquad \xi_{\gamma\mu\nu} = -(x_{\mu}\delta^{\nu}_{\gamma} - x_{\nu}\delta^{\mu}_{\gamma}); \tag{13}$$

(iii) dilation

$$\tau^{\alpha} = t^{\alpha}, \qquad \xi_{\alpha} = -x_{\alpha}; \tag{14}$$

(iv) inversions

$$\tau^{\alpha}_{\lambda} = 2t^{\alpha} t_{\lambda} - t^{\nu} t_{\nu} \delta^{\alpha}_{\lambda}, \qquad \xi_{\gamma\lambda} = -2x_{\sigma}(t^{\sigma}\delta^{\gamma}_{\lambda} + t^{\lambda}\delta^{\sigma}_{\gamma} - t^{\gamma}\delta^{\sigma}_{\lambda}); \tag{15}$$

where $\mu < \nu$ in (13).

3. Conformal identities

We are now in a position to write down a set of conformal identities for the variational integral (1) when (1) is invariant under the special conformal transformations (8)–(11). The identities that we obtain will represent a set of fifteen quasilinear first-order partial differential equations in the Lagrangian L and will be necessary and sufficient for (1) to be conformally invariant.

Under the translations (8), it immediately follows from (3) and (12) that

$$\frac{\partial L}{\partial t^{\alpha}} = 0, \qquad \alpha = 1, \ldots, 4. \tag{16}$$

These four equations state that the Lagrangian cannot depend explicitly upon the space–time coordinates $t^1, \ldots, t^4$.

For the dilation, or scale transformation, (10), it can be seen from (14) that

$$\frac{\mathrm{d}\xi_{\gamma}}{\mathrm{d}t^{\alpha}} = -\frac{\partial x_{\gamma}}{\partial t^{\alpha}}, \qquad \frac{\mathrm{d}^2\tau_{\gamma}}{\mathrm{d}t^{\alpha}\,\mathrm{d}t^{\beta}} = 0 \qquad \text{and} \qquad \frac{\mathrm{d}^2\xi_{\gamma}}{\mathrm{d}t^{\alpha}\,\mathrm{d}t^{\beta}} = -\frac{\partial^2 x_{\gamma}}{\partial t^{\alpha}\,\partial t^{\beta}}.$$

Substituting into (3) and using (16), we obtain the single identity

$$4L = 2\frac{\partial L}{\partial(\partial_{\alpha} x_{\gamma})}\partial_{\alpha} x_{\gamma} + 3\frac{\partial L}{\partial(\partial_{\alpha\beta} x_{\gamma})}\partial_{\alpha\beta} x_{\gamma} + \frac{\partial L}{\partial x_{\alpha}} x_{\alpha}. \tag{17}$$

We shall comment further on (17) in the sequel.

There are six identities obtained from the rotations (9). From (13) we get, with $\mu < \nu$,

$$\frac{\mathrm{d}\tau^{\gamma}_{\mu\nu}}{\mathrm{d}t^{\beta}} = \delta^{\gamma}_{\mu}\delta^{\nu}_{\beta} - \delta^{\gamma}_{\nu}\delta^{\mu}_{\beta}, \qquad \frac{\mathrm{d}^2\tau^{\gamma}_{\mu\nu}}{\mathrm{d}t^{\alpha}\,\mathrm{d}t^{\beta}} = 0$$

$$\frac{\mathrm{d}\xi_{\gamma\mu\nu}}{\mathrm{d}t^{\beta}} = -(\partial_{\beta} x_{\mu}\delta^{\nu}_{\gamma} - \partial_{\beta} x_{\nu}\delta^{\mu}_{\gamma}), \qquad \frac{\mathrm{d}^2\xi_{\gamma\mu\nu}}{\mathrm{d}t^{\alpha}\,\mathrm{d}t^{\beta}} = -(\partial_{\alpha\beta} x_{\mu}\delta^{\nu}_{\gamma} - \partial_{\alpha\beta} x_{\nu}\delta^{\mu}_{\gamma}).$$

Substitution into (3) yields

$$M_{\mu\nu} + N_{\mu\nu} = 0, \tag{18}$$

where,

$$M_{\mu\nu} \equiv -\frac{\partial L}{\partial(\partial_\nu x_\alpha)}\partial_\mu x_\alpha + \frac{\partial L}{\partial(\partial_\mu x_\alpha)}\partial_\nu x_\alpha - \frac{\partial L}{\partial(\partial_{\nu\beta} x_\gamma)}\partial_{\mu\beta} x_\gamma$$
$$-\frac{\partial L}{\partial(\partial_{\mu\beta} x_\gamma)}\partial_{\nu\beta} x_\gamma + \frac{\partial L}{\partial(\partial_{\beta\nu} x_\gamma)}\partial_{\beta\mu} x_\gamma - \frac{\partial L}{\partial(\partial_{\beta\mu} x_\gamma)}\partial_{\beta\nu} x_\gamma \tag{19}$$

and

$$N_{\mu\nu} \equiv \frac{\partial L}{\partial x_\mu} x_\nu - \frac{\partial L}{\partial x_\nu} x_\mu + \frac{\partial L}{\partial(\partial_\alpha x_\mu)}\partial_\alpha x_\nu - \frac{\partial L}{\partial(\partial_\alpha x_\nu)}\partial_\alpha x_\mu + \frac{\partial L}{\partial(\partial_{\alpha\beta} x_\mu)}\partial_{\alpha\beta} x_\nu - \frac{\partial L}{\partial(\partial_{\alpha\beta} x_\nu)}\partial_{\alpha\beta} x_\mu. \tag{20}$$

We note that $N_{\mu\nu}$ and $M_{\mu\nu}$ are antisymmetric in their indices.

In the case of inversions, it follows from (15) that

$$\frac{d\tau_\lambda^\alpha}{dt^\beta} = 2(t^\alpha\delta_\beta^\lambda + t^\lambda\delta_\beta^\alpha - t^\beta\delta_\lambda^\alpha), \qquad \frac{d^2\tau_\lambda^\alpha}{dt^\beta\, dt^\gamma} = 2(\delta_\gamma^\lambda\delta_\beta^\alpha + \delta_\gamma^\alpha\delta_\beta^\lambda - \delta_\gamma^\beta\delta_\lambda^\alpha)$$

$$\frac{d\xi_{\gamma\lambda}}{dt^\alpha} = -2\partial_\alpha x_\sigma(t^\sigma\delta_\lambda^\gamma + t^\lambda\delta_\gamma^\sigma - t^\gamma\delta_\lambda^\sigma) - 2(x_\alpha\delta_\lambda^\gamma + x_\gamma\delta_\alpha^\lambda - x^\lambda\delta_\alpha^\gamma)$$

$$\frac{d^2\xi_{\gamma\lambda}}{dt^\alpha\, dt^\beta} = -2(\partial_\beta x_\alpha\delta_\lambda^\gamma + \partial_\beta x_\gamma\delta_\alpha^\lambda - \partial_\beta x_\lambda\delta_\alpha^\gamma - \partial_\alpha x_\beta\delta_\lambda^\gamma - \partial_\alpha x_\gamma\delta_\beta^\lambda + \partial_\alpha x_\lambda\delta_\beta^\gamma)$$
$$-2\partial_{\alpha\beta} x_\sigma(t^\sigma\delta_\lambda^\gamma + t^\lambda\delta_\gamma^\sigma - t^\gamma\delta_\lambda^\sigma).$$

When these expressions are substituted into (3) and extensive simplifications are made, we obtain finally

$$Q_\lambda \equiv \frac{\partial L}{\partial(\partial_\alpha x_\alpha)} x_\lambda + \frac{\partial L}{\partial(\partial_{\alpha\alpha} x_\beta)}\partial_\lambda x_\beta + \left(\frac{\partial L}{\partial(\partial_{\alpha\beta} x_\beta)} + \frac{\partial L}{\partial(\partial_{\beta\alpha} x_\beta)}\right)\partial_\alpha x_\lambda - \left(\frac{\partial L}{\partial(\partial_{\alpha\beta} x_\lambda)} + \frac{\partial L}{\partial(\partial_{\beta\alpha} x_\lambda)}\right.$$
$$\left. + 2\frac{\partial L}{\partial(\partial_{\alpha\lambda} x_\beta)} + 2\frac{\partial L}{\partial(\partial_{\lambda\alpha} x_\beta)}\right)\partial_\alpha x_\beta - \left(\frac{\partial L}{\partial(\partial_\beta x_\lambda)} + \frac{\partial L}{\partial(\partial_\lambda x_\beta)}\right) x_\beta = 0. \tag{21}$$

We summarise the results in the following theorem.

Theorem 2. Let the action integral J be given by (1) where $x_1(t), \ldots, x_4(t)$ are components of a covariant vector field. If J is invariant under the special conformal group defined by (8)–(11), then the Lagrangian L must satisfy the following fifteen identities:

(*a*) $$\frac{\partial L}{\partial t^\alpha} = 0, \qquad \alpha = 1, \ldots, 4 \tag{22}$$

(*b*) $$4L = \frac{\partial L}{\partial x_\alpha} x_\alpha + 2\frac{\partial L}{\partial(\partial_\alpha x_\gamma)}\partial_\alpha x_\gamma + 3\frac{\partial L}{\partial(\partial_{\alpha\beta} x_\gamma)}\partial_{\alpha\beta} x_\gamma \tag{23}$$

(*c*) $$N_{\mu\nu} + M_{\mu\nu} = 0 \qquad (\mu, \nu) \in S \tag{24}$$

(*d*) $$Q_\lambda = 0, \qquad \lambda = 1, \ldots, 4. \tag{25}$$

Conversely, if (22)–(25) hold, then (1) is invariant under (8)–(11).

Besides providing a test to check whether or not a given action integral is conformally invariant, equation (23) implies the existence of homogeneity conditions in special circumstances.

Corollary. If, in addition to the hypotheses on the theorem, the Lagrangian $L \equiv L(\partial^2 x)$ depends *only* upon second derivatives of the field functions, then L must be homogeneous of degree 4/3; i.e.

$$L(k\partial^2 x) = k^{4/3} L(\partial^2 x), \qquad k > 0.$$

The proof follows directly from Euler's theorem on homogeneous functions.

In theory, but with difficulty in practice, theorem 2 also provides conditions which can be used to characterise *all* Lagrangians whose associated variational problem is conformally invariant. This can be accomplished by regarding equations (22)–(25) as a system of partial differential equations in the unknown Lagrangian L and using the method of characteristics to obtain information about L. More practical, however, is to use equations (22)–(25) as a test for conformal invariance of a given Lagrangian, as we now illustrate in the next section. We emphasise that equations (22)–(25) are neither conservation laws nor Noether identities; rather, they are invariance identities which involve only second-order derivatives of the field functions, in contrast to the Noether identities which involve fourth-order derivatives.

4. Application to generalised electrodynamics

We conclude by showing how the above results can be applied to what is usually known as 'generalised electrodynamics' (see Podolsky and Schwed 1948). This theory, which was developed to eliminate the infinities associated with a point source, is characterised by a second-order Lagrangian

$$\begin{aligned} L &= \frac{1}{2}(\boldsymbol{E}^2 - \boldsymbol{H}^2) + \frac{a^2}{2}\left[(\operatorname{div}\boldsymbol{E})^2 - \left(\operatorname{curl}\boldsymbol{H} - \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t}\right)^2\right] \\ &= -\tfrac{1}{2}[\tfrac{1}{2}F_{\alpha\beta}^2 + a^2(\partial_\beta F_{\alpha\beta})^2], \end{aligned} \tag{26}$$

where a is a constant and

$$F_{\alpha\beta} \equiv \partial_\alpha x_\beta - \partial_\beta x_\alpha$$

is the electromagnetic field tensor, where $x_1, \ldots, x_4$ are the components of the four-potential; the Lagrangian is clearly second order through the appearance of the terms $\partial_\beta F_{\alpha\beta}$ which contain second derivatives of the field functions $x_1, \ldots, x_4$.

It is well known (see Bessel-Hagen 1921) that the standard action integral

$$J = \iiiint -\tfrac{1}{4}F_{\alpha\beta}^2 \, dt^1 \ldots dt^4 \tag{27}$$

for the electromagnetic field *in vacuo* is invariant under the special conformal group. Using the results of § 3 it is possible to show that generalised electrodynamics is *not* conformally invariant unless the constant $a = 0$,which is the classical case.

To make this observation, we first calculate the various derivatives of L, where L is given by (26). With a little effort, it follows that:

$$\frac{\partial L}{\partial(\partial_\alpha x_\beta)} = -(\partial_\alpha x_\beta - \partial_\beta x_\alpha) \tag{28}$$

and

$$\frac{\partial L}{\partial(\partial_{\mu\nu} x_\gamma)} = -a^2(\partial_{\mu\beta} x_\beta \delta^\nu_\gamma - \delta_{\gamma\beta} x_\beta \delta^\nu_\mu - \partial_{\beta\beta} x_\mu \delta^\nu_\gamma + \partial_{\beta\beta} x_\gamma \delta^\mu_\nu). \tag{29}$$

Therefore

$$\frac{\partial L}{\partial(\partial_\alpha x_\beta)} \partial_\alpha x_\beta = -[(\partial_\alpha x_\beta)^2 - (\partial_\alpha x_\beta)(\partial_\beta x_\alpha)] \tag{30}$$

and

$$\frac{\partial L}{\partial(\partial_{\mu\nu} x_\gamma)} \partial_{\mu\nu} x_\gamma = -a^2(\partial_{\gamma\beta} x_\beta - \partial_{\beta\beta} x_\gamma)^2. \tag{31}$$

By substituting (28)–(31) into (23), we observe that (23) reduces to an identity only in the case $a = 0$. This fact, along with the Bessel-Hagen theorem, implies that

$$J = \int\int\int\int L \, dt^1 \, dt^2 \, dt^3 \, dt^4$$

is conformally invariant, where L is given by (26), if, and only if, $a = 0$.

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